SEMIPRIME PURELY NON-ASSOCIATIVE ACCESSIBLE RINGS

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ABSTRACT:

Thedy [1] proved the result for prime right alternative and free of locally nilpotent-ideals. Kleinfeld [2] proved that if a prime alternative ring is not associative then its nucleus N equals its center C.

In this paper we investigate the results of accessible ring. First we prove that if R is semiprime and purely non-associative, then N = C. Also we prove that middle nucleus=center of R if R is purely non-associative provided that either R has no locally nilpotent ideals or R is semiprime and finitely generated by mod M.

Key Words:
Accessible ring, Semiprime Ring, Commutator, Associator, Nucleus, Center, Characteristic.

Introduction:

An accessible algebra R of characteristic $\neq 2$ is semiprime if there exist a non-zero ideal I such that $I^2 = (0)$.

In any non-associative algebra R, the commutator (a, b) and associator (a, b, c) are defined by $ab - ba$ and $(ab)c - a(bc)$.The algebra is said to be of characteristic$\neq$2 if $2a = 0$ implies $a = 0$ for a belongs to R and throughout this paper R is assumed to be accessible ring of characteristic$\neq$2.

The right Nucleus M, the nucleus N and the center C are defined by

$$M = \{m \in R: (R, R, m) = 0\}$$
$$N = \{n \in R: (n, R, R) = 0\}$$
$$C = \{C \in N: (C, N) = 0\}$$

In [3] Thedy was proved that

$$(M, R) \subseteq M \text{ and } (M, R, R) \subseteq M$$

MAIN RESULTS:

LEMMA 1: Suppose that $m \in M$ and $x, y, z \in R$ then

(i) $(\mathrm{x}, \mathrm{y}, \mathrm{zm}) = (\mathrm{x}, \mathrm{y}, \mathrm{z})\mathrm{m}$, $(\mathrm{mx}, \mathrm{y}, \mathrm{z}) = \mathrm{m}(\mathrm{x}, \mathrm{y}, \mathrm{z})$

(ii)$(\mathrm{xy}, \mathrm{m}) = \mathrm{x}(\mathrm{y}, \mathrm{m}) + (\mathrm{x}, \mathrm{m})\mathrm{y}$

(iii)$(\mathrm{x}, \mathrm{y}, \mathrm{z})(\mathrm{m}, \mathrm{z}) = 0$

(iv) $(\mathrm{x}, \mathrm{y}, \mathrm{z})(\mathrm{m}, \mathrm{w}, \mathrm{z}) = 0$

and (v) If $(\mathrm{m}, \mathrm{R}) = 0$ then $m = C$.

PROOF:

(i) The Teichmular identity we have

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y) \quad (1)$$

Put $w = m$ in (1) gives

$$(mx, y, z) - (m, xy, z) + (m, x, yz) = m(x, y, z) + (m, x, y)$$

this implies $(mx, y, z) = m(x, y, z)$

Similarly we have $(x, y, zm) = (x, y, z)m$

(ii) The semi-Jacobi identity

$$(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, y, z) \quad (2)$$

In accessible ring (2) becomes

$$(xy, z) = x(y, z) + (x, z)y \quad (3)$$

Put $z = m$ in (3) gives

$$(xy, m) = x(y, m) + (x, m)y.$$

(iii) $(z^2, m) = z(z, m) + (z, m)z$ from(ii)

$$= 2\mathrm{z}(\mathrm{z}, \mathrm{m}) - (\mathrm{z}, (\mathrm{z}, \mathrm{m}))$$

Thus $\mathrm{z}(\mathrm{z}, \mathrm{m}) \in \mathrm{M}$.

And by part (i) $(\mathrm{x}, \mathrm{y}, \mathrm{z})(\mathrm{m}, \mathrm{z}) = (\mathrm{x}, \mathrm{y}, \mathrm{z}(\mathrm{m}, \mathrm{z})) = 0$.

(iv) $(x, y, z)(m, w, z) = \big(x, y, z(m, w, z)\big)$

$$= \big(x, y, (mz, w, z)\big)$$
$$= 0.$$

(V) From $((z, x), y) + ((x, y), z) + ((y, z), x) = 2(x, y, z) + 2(y, z, x) + 2(z, x, y)$

Put $x = m$, in above equation gives

$$(m, y, z) = 0.$$

Hence all the results are proved.♦

Let $\bar{R}$ be the ring obtained by adjoining 1 to R in the usual way.

LEMMA 2: If $\mathrm{n} \in \mathrm{N}$, then the ideal of R generated by (R, n) is

$V_n = \bar{R}(R, n) = (R, n)\bar{R}$.

PROOF: Here $\bar{R}(R, n)$ is the set of all finite sums $\sum(r_i, n) + \sum s_j(t_j, n)$.

From Lemma (1(ii)) gives

$$(\mathrm{xy}, \mathrm{n}) = \mathrm{x}(\mathrm{y}, \mathrm{n}) + (\mathrm{x}, \mathrm{n})\mathrm{y}.$$

So that the two expressions for V_n are equal.

Then,

$R\,.\,V_n = R.\bar{R}(R,n) = \bar{R}(R,n) \subseteq V_n$

$V_n.\,R = \bar{R}(R,n).\,R \subset R\,.\,V_n \subseteq V_n$.

Hence Lemma proved. ♦

LEMMA 3: Let V be the ideal of R generated by (R, M) and let

$P = \{p \in R : pv = 0\}$ then

(i) $V = \bar{R}(R,M) = (R,M)\bar{R}$

(ii) If $p(M,R) = 0$ then $p \in P$.

(iii) P is an ideal of R.

PROOF: The identity

$x(y,m) = (x,m') + (y,m)x$ for $m' = (y,m) \in M$.

and it proves $\bar{R}(R,M) = (R,M)\bar{R}$.

clearly, $\bar{R}(R,M)$ is a left ideal, and

$\bar{R}(R,M)\,.\,R \subseteq \bar{R}\bar{R}(R,M) = \bar{R}(R,M)$.

Which shows that it is two -sided.

(ii) If $p(M,R) = 0$, and then $pV = p(M,R)\,\bar{R} = 0$

(iii) If $p \in P$ and $r \in R$, then

$pr.\,(M,R) \subseteq pV = 0$

and $rp.\,(M,R) = 0$

Therefore p is an ideal of R.♦

LEMMA 4: Suppose that R is semiprime and purely non-associative, then for all $m,n \in M$ and $x,y \in R$ we have

$(i)\ (m,n)2 = 0$

$(ii)\ (m,n) = 0$

$(iii)\ (x,n)(x,m) = 0$

$(iv)\ (x,m)(y,m) = 0$

PROOF:

We set $W = \{r \in m / rR \subseteq m\}$ and $P = \{p \in R / pW = 0\}$

In [4] it was shown that P and W are ideals of R with $(R,R,R) \subseteq P$.

For the accessible ring we have $(R,R,R) \subseteq P$

Now $(P \cap W)^2 \subseteq PW = 0$.

and semi primeness give $P \cap W = 0$

Since $(W,R,R) \subseteq P \cap W$ we find $W \subseteq N$

Hence by pure non-associativity W=0

Now for $(m,x)^2 \in W$ and $(M,M) \subseteq W$.

Thus we have (i) & (ii).

(iii) Linearizing (i) on m, we have

$(m,x)(n,x) + (n,x)(m,x) = 0$

Since M is commutative by (ii),

This gives $2(m,x)(n,x) = 0$

For characteristic $\neq 2$ which implies $(m,x)(n,x) = 0$.

(iv) Linearize (i) on x ,we have

$(m,x)(m,y) + (m,y)(m,x) = 0$

Since M is commutative by (ii)

Thus gives $2(m,x)(m,y) = 0$

For characteristic $\neq 2$, $(m,x)(m,y) = 0$.♦

THEOREM 1: Suppose that R is semiprime and purely non-associative then $N = C$.

PROOF: Given $n \in N$, let V_n be as in Lemma (2).

Then $V_n^2 = \bar{R}\,(R,n)\,.\,(R,n)\,\bar{R}$

$= \bar{R}\,(R,n)^2\,\bar{R}$

$= 0$ (By Lemma (4 (iv)))

By semiprimeness $V_n = 0$, whence $n \in C$.

Thus $N \subset C$, so $N = C$.♦

COROLLARY 1: Suppose that R is prime but non-associtative then $N = C$.

PROOF: It is sufficient to show that R is purely non-associative.

Let I be an ideal in the Nucleus.

Then $(R,R,R)I = (R,R,RI) \subseteq (R,R,I) = 0$

Thus if $A = \bar{R}(R,R,R)$ is the associtative ideal of R, then $AI = (0)$,

But R is non-associative and prime, so $I = (0)$.

LEMMA 5: If $m \in M$ and $m(M,R) = 0$ then , $m \in C$.

If further $m^2 = 0$ then $m = 0$.

PROOF: Let $P = \{p \in R \mid pV = 0\}$.

Then $m \in P$ (By lemma 3(ii)).

So $(m,R) \subseteq p \cap V$.

Since $pV = 0$ then by lemma (4) that $(m,R) = 0$.

So $m \in C$ (By lemma 1(V)).

Hence the ideal generated by m is $\bar{R}m$.

If $m^2 = 0$ then $(\bar{R}m)^2 = 0$.

By semiprimeness $\bar{R}m = 0$ and $m = 0$.♦

For a given finite list $A = \{a_1, a_2, \ldots, a_k\}$ of elements of R, define $T(A) = (M,a_1)(M,a_2)\ldots(M,a_k)$ that is $\{(m_1,a_1)(m_2,a_2)\ldots(m_k,a_k)\} : m_i \in M\}$.

Note that $T(A) \subseteq M$. Also, by lemma (4(ii)), $T(A)$ does not depend on the order of the a_i, and by lemma (4(iii)) it is 0 If A has any repetitions. For the same reason if $t \in T(A)$ then $t^2 = 0$.

We allow the empty list $A = \emptyset$, defining $T(\emptyset) = 1$(the unit element of $\bar{R}$). It may be checked that $(M,a)T(A) = T\{A \cup \{a\}\}$ in all cases including $A = \emptyset$□.

Next define $L(A) = \{w \in R : (w,M)T(A) = 0\}$.

In particular, $L(\emptyset)=\{w\in R:(w,M)=0\}$.

LEMMA 6: (i) If $b\in L(A)$ then $(M,b,R)T(A)=0$.
(ii) $L(A)$ is subring of R.

PROOF: (i) we have

$$0 = (M,R,R)(b,M)T$$
$$= (M,b,R)(R,M)T \quad \text{(by lemma 1(iii))}$$
$$= (M,b,R)T(R,M) \quad \text{(by lemma 4(ii))}$$
$$= (M,b,R)(R,M) \quad \text{(by lemma 1(i))}$$

If $z\in(M,b,RT)$ Then $z(R,M)=0$.

Also z is of the form (m,b,r), so that $z^2=0$ (by Lemma 1 (iv))

Hence $z=0$ (by Lemma 5)

i.e. $0=(M,b,RT)=(M,b,R)T$.

(ii) Suppose that $x,y\in L(A)$ and $m\in M$

Then

$$(xy,m) = x(y,m)+(x,m)y \quad \text{(by Lemma 1(ii))}$$
$$= x(y,m)+(m',y)+y(x,m)$$

Where $m'=(x,m)\in M$

Since $T\subseteq M$ we now have

$$(xy,m)T \subseteq x(y,m)T+(m',y)T+y(x,m)T$$

The R.H.S of above is 0 by assumption.

Since $m\in M$ was arbitrary this shows that $(xy,M)T=0$, so that $xy\in L$.

Let us say that R is finitely generated $mod\ M$ of there is a finite subset A of R such that the subring of R generated by $M\cup A$ is all of R.♦

THEOREM 2: Suppose that R is semiprime purely non-associative and is finitely generated by $mod\ M$ then $M=C$.

PROOF: Suppose that R is generated by $M\cup A$.

Where $A=\{a_1,a_2,\ldots,a_k\}$, we will show that if S is any list of terms from A then $L(s)=R$ and provided that $S\neq\emptyset$, $T(s)=0$.

We prove this by reverse induction on the length $r=|s|$ of S.

If $|s|=k+1$ then S has a repetition, so that $T(s)=0$,

Hence clearly $L(s)=R$.

Suppose we have both results for lists of length $r+1$, and S is a list of length r. Then for $a\in A$ we have $(a,M)T(s)=T(s')$,

Where $s'=s\cup\{a\}$ has length $r+1$.

Thus $(a,M)T(s)=0$.So that $\in L(s)$.

Hence $A\subseteq L(s)$, as $(M,M)=0$ (By lemma 4(ii)).

We also have $M\subseteq L(s)$.

Thus by lemma (6(ii)), $L(s)$ is a subring of R contained $A\cup M$,

$i.e\ L(s)=R$.

Next suppose that $S\neq\emptyset$, and $t\in T(s)$.

Since $L(S)=R$, we have $(R,M)T(s)=0$.

So that $t(R,M)=0$.

Also we seen that $t^2=0$.

So by Lemma (5) we have $t=0$.

i.e., $T(s)=0$

Finally the result $L(\emptyset)=R$ gives $(R,M)=0$.

Hence $M=C$ by Lemma (1(V)).♦

THEOREM 3: Suppose that R is purely non-associative and free of locally nilpotent ideals. Then $M=C$.

PROOF: By lemma (4(ii)), M is commutative.

If we let I be the nil radical of M,

Then from [1], $I+IR$ is locally nilpotent ideal of R such that $(M,R,R)(M,R)\subseteq I$.

Since R is free of locally nilpotent ideals $(M,R,R)=0$ and $(M,R)=0$.

Hence $M=C$ by Lemma 1(v).♦

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